Vibrational Analysis and Modeling of Skyscraper Response to Earthquakes

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I. Background

Although skyscrapers as they appear today are a relatively recent achievement, the motivation to build up into the sky has been in the human consciousness for some time.Tall structures, like towers, were first made from stone, but stone was heavy and structurally incapable of accommodating windows for lighting.[2] Developments in engineering led to such structures as the flying buttress, which helped to deliver some of the loads imposed on the structures to the ground and allowedfor increased flexibility in construction.[2] Experimentation with concrete during the nineteenth century assisted in the development of skyscrapers; differing compositions of concrete have a variety of structural abilities, drying times, and appearances.[1] Soon thereafter, steel became the primary material for use in the construction of buildings, but it wasn't until Robert Talbot published a paper about the possibility ofsteel beams being reinforced with concrete that the skyscraper really began to develop.[1]

With the use of concrete and steel in tandem came better structural security and an increased ability to hold and transfer loads. These abilities also led to the development by many, most notably Fazlur Khan, of new structural designs.Khan, a Bangladeshi born and primarily American educated structural engineer, played a pivotal role in the development of the modern skyscraper.[5]He aided

Figure 1: Fazlur Khan with his daughter, Yasmin [5]

in the development of the Frame-Shear Wall structure and is credited with the invention of the Tubular structure.[1]

The frame-shear wall structure operates by using the natural movement of the frame and the shear wall to restrict movement.[8] Independently, the frame structure and the shear wall will move in opposite directions when a force is present.[8]As shown in Figure 2, when the frame and shear wall are used in tandem, each piece exhibits the same behavior as it did independently, but the juxtaposition of the structures allows them to push against each other and opposeeach other's motion.[8]

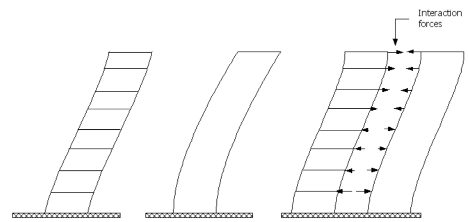

Figure 2: Frame-shear Wall structure [8]

In the tubular structure, a building is divided up into sections, each of which is made up of vertical steel columns.[10]These columns are arranged in such a way as to form multiple square tube shapes within the structure of the building.[10] Figure 3 shows the cross-sectional views of the skyscraper at several levels, each of which resembles a grid.[10]Structurally, the effect of the tubes is anincreased ability to resist lateral forces, which are typically caused by wind.[1] This is done in much the same way as with the frame-shear wall structure.[1] The combined outlinesof the tubes work together to resist movement by pushing on each other.[1]To continue the idea of structural support, certain areas within the structure contain extra horizontal connections

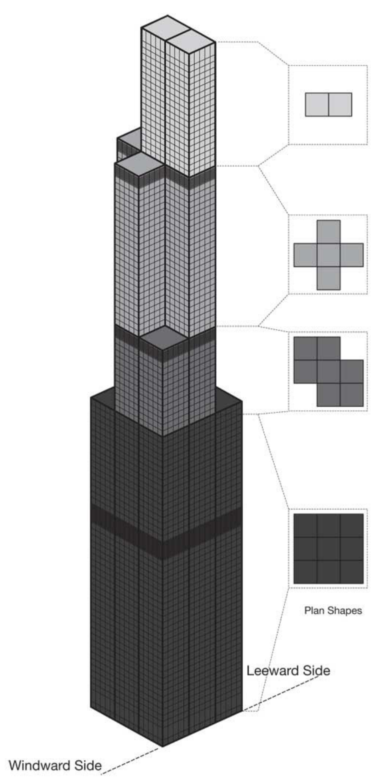

Figure 3: Tubular structure in the Willis Tower (formerly the

(represented by the darkened bands in Figure 3) in order to support the heating, ventilation, and air conditioning systems as well as to distribute the total load imposed by the above floors among the beams.[10] Many buildings with some variation of the tubular structure also feature inset sections as part of their design. This structure reduces the weight of the building at the top, allowing for more flexibility in the wind and lower load support requirements below. This design is currently the preferred design in the construction of new skyscrapers.

One of the most important things a building must do is withstand the force of an earthquake. During an earthquake, the ground that a building sits on shifts, pulling the base of the building with it.[4] Due to inertia, the top of the building remains motionless until the force of the earthquake reaches it.[4] The building begins to act like a wave; the constantly shifting ground acts as a force of vibration and the rest of the building is allowed to vibrate freely. The amplitude of the vibration at the top of the building depends on its length. Longer buildings, like skyscrapers, are more flexible than mid-length buildings,[3] and so have longer amplitudes. Additionally, long buildings have longer periods of vibrations, which combines with the long amplitude to reduce the immediate stress put on a tall building versus a short building.

Even though the stress put on a skyscraper by earthquake vibrations may be smaller than those put on a small high-rise building, it is still important that these vibrations be managed. Too much vibration, be it in the scope of force or time, will damage any building and become a danger to the people in and around it.Fortunately, there are multiple ways to improve the strength of a building.

Part of the strength of a building comes from the materials that make up its structure. For instance, steel is preferred over wood for use in tall buildings because it is stronger and requires more force to be applied to it before it will deform. Steel is also more fire resistant than wood, though neither material is particularly strong under heat. In that regard, concrete is a much more reliable choice of material.[1] Concrete is also at optimal strength when being compressed[1] but deforms when under tension, which means concrete can support significant loads without cracking. Three of the additional benefits of concrete are that it dampens vibrations that pass through it, its production is less expensive than that of steel, and its production can be started before a project officially begins via the use of formwork[1]. Formwork is a type of mold that concrete can be poured into so that it dries in a particular shape.[1] These shapes can then be connected to form a structure.[1] This type of concrete construction is often used in parking garages; large slabs are constructed before the project construction begins. When the structure is ready for concrete, the slabs are shipped to the worksite and then are lifted into place and bonded with more concrete. This makes the formation of particular shapes much more practical and allows commonly used shapes to be readily available for any project.

Another way to improve the strength of a building is through the use of its structure. This can be shown with the use of basic shapes in the frame of the building. Often, reinforcing bars, called knee braces, are placed diagonally in the corners of rectangular framework as shown in Figure 4 to prevent rotation about the connection points between trusses and columns. The idea of structural strength can also be used on the large scale. The orientation and size of the framework in the structure can determine the amount of force that can be applied to the frame before it becomes

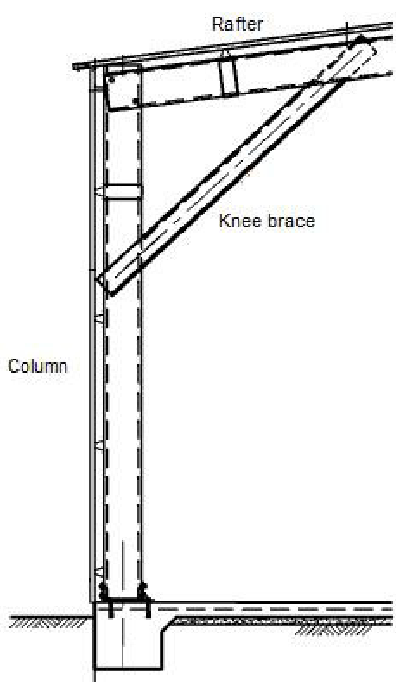

Figure 4: Knee bracing in corner structure [9]

damaged. For example, a piece of cardboard is much weaker along its smooth, flat sides than against its cut, corrugated edges. Therefore, it will require more force to bend the cardboard when that force is applied to the corrugation than when it is applied to the flat side. Although this is a simplistic model, the same concept is applicable to many materials in a variety of shapes, steel beams included.

One of the most common ways to manage the vibrational force exerted on a building is through the use of a damping device. These devices reduce the force felt by the building from an earthquake in a variety of ways and can be placed into two categories. These categories are material based and additional mass.[6]Material based dampers include friction, electro-magnetic, hysteric, viscous, and viscoelastic dampers.[6] These dampers work to reduce the motion of the building by transferring the kinetic energy of the earthquake to some other type of energy which then disperses from the system.[4]Friction dampers disperse energy by transferring the kinetic energy of the building and the ground into heat.[4] Viscoelastic dampers control the breaking of solid materials and Viscous dampers, like the piston shown in Figure 5, use forced fluid movement to counteract motion.[4] These types of dampers are often placed in locations that will minimize the movement of the structure. In this way, they can also function as bracing units.[6]

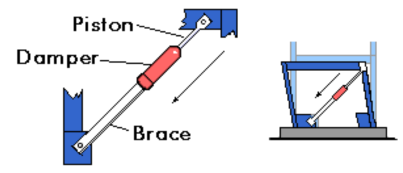

Figure 5: Viscous damper acting in a bracing system [4]

Additional mass dampers can be further broken down into tuned mass dampers (TMD) and tuned liquid dampers (TLD).[6]Tuned mass dampers tend to embody the idea of a pendulum; the damper

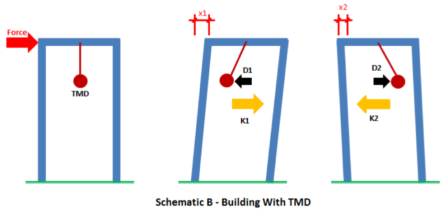

Figure 6: Tuned Mass Damper [7]

must be as free to move as possible in order to restrict vibration. These dampers are typically placed in the top floors of a building and utilize inertia to reduce vibrational forces.[12]When the building shakes, the pendulum stays where it is for a longer period of time than the top floors do. This is becausethe force of the earthquake on the building must move up from the base of the building to the top floors of the building and then to the damper itself. When the pendulum does begin to move, it will be opposite the upper floors of the building, just as is shown in Figure 6. Due to the contrasting motion, the upper floors and the pendulum will pull on each other. The contrary motion will make the net force on the top floors be very close to zero, which means that area of the building will stop moving.

In contrast, tuned liquid dampers have a bit more flexibility. As Figure 7 portrays, the additional mass is a liquid, so it can be contained in a unit and be allowed to move freely within the unit.[6] This liquid also tends to be water, simply because of the convenience and multiple uses of water. If a building's water storage rests on its roof, then the storage unit doubles as a tuned liquid damper as it will allow the water to move freely and exhibit inertial properties within the container.[6] As such, the shape of the storage unit can also determine the period of oscillation of the water; the period will vary based on the size of a rectangular tank.[6]

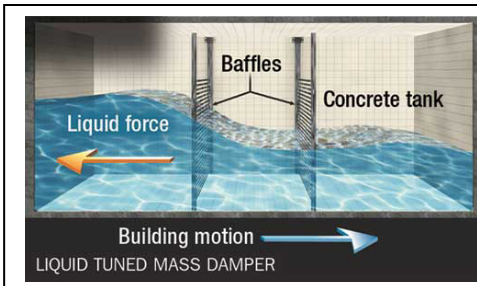

Figure 7: Tuned Liquid Damper [11]

II. Two-Degree of Freedom (Model 1)

The first simulation shows a two-degree of freedom model. These degrees of freedom are linear movement in the vertical direction and rotation about the center of mass of the floor; horizontal movement is not included. The simulation begins with an arbitrary"floor" from a theoretical building, like that shown in Figure 8, being displaced by some distance, y. In its displaced position, the sum of forces acting on the floor is given by $\sum F = ma_y = m\ddot{y} = F_L + F_R - mg$ and the sum of torques acting on the floor by $\sum M_{cg} = I\alpha = I\ddot{\theta} = (F_R - F_L)\left(\frac{D}{2}\right)$. In this model,

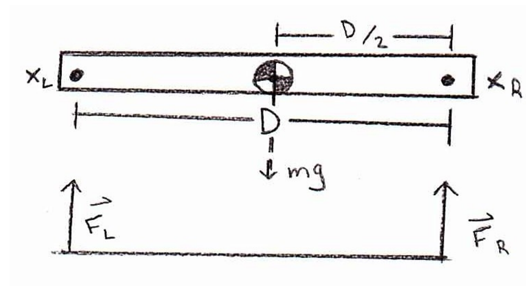

Figure 8: Forces and lengths associated with the floor when it is not displaced

the angle of rotation will be a changing value and will need to be written in terms of the given constants. Based upon Figure 9, $y = \frac{(x_1 + x_2)}{2}$ and $\dot{y} = \frac{(x_1 + x_2)}{2}$, where $\dot{y}$ is synonymous with y'. Figure 9 also shows that $\sin\phi = \frac{(x_2 - x_1)}{D}$.In addition, the Taylor series representation of sine is given by $\sin\phi = \phi + \frac{\phi^3}{3!} + \ldots$, and can be truncated to $\sin\phi = \phi$. Small angle approximation

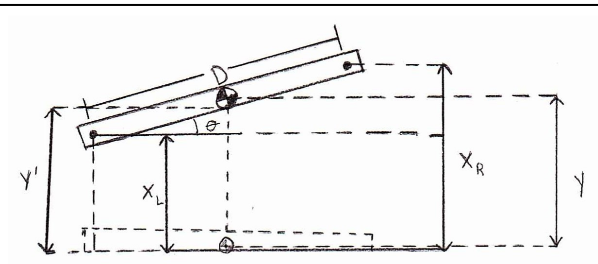

Figure 9: Lengths associated with the floor when the floor is displaced

using those three pieces shows that so long as phi is less than 6 degrees, the error contributionof truncating the expression for sine in the simulation will be .01%.Since sine is considered to be equivalent to phi, phi can be restated as $\phi = \frac{(x_2 - x_1)}{D}$ and its second derivative as $\ddot{\phi} = \frac{\left(\ddot{x}_2 - \ddot{x}_1\right)}{D}$. Since $y = \frac{(x_1 + x_2)}{2}$, the second derivative of y is $\ddot{y} = \frac{\left(\ddot{x}_1 + \ddot{x}_2\right)}{2}$.

Substituting the expression for $\ddot{y}$ into that of the sum of forces acting on the floor gives

$\frac{m\left(\ddot{x}_1+\ddot{x}_2\right)}{2}=F_R+F_L-mg$. Rearranging this equation to solve for $\ddot{x}_1$ and $\ddot{x}_2$ gives

$\ddot{x}_1+\ddot{x}_2=\frac{2}{m}\{F_R-F_L\}-\frac{g}{2}$, which will be referred to as Equation I.In the same fashion, $\ddot{y}$ can be

substituted into the equation for the sum of torques acting on the floor, leading to

$I\ddot{\theta}=I\frac{\left(\ddot{x}_2-\ddot{x}_1\right)}{D}=\left(F_R-F_L\right)\left(\frac{D}{2}\right)$. This can be rewritten as $\ddot{x}_2-\ddot{x}_1=\frac{D^2}{2I}\left(F_R-F_L\right)$ and will be

called Equation II.

The expressions for $\ddot{x}_1$ and $\ddot{x}_2$ can be found by utilizing Equations I and II in a system of

equations. Solving for these values gives $\ddot{x}_1=\frac{1}{m}\{F_R+F_L\}-\frac{g}{4}-\frac{D^2}{4I}\left(F_R-F_L\right)$ and

$\ddot{x}_2=\frac{1}{m}\{F_R+F_L\}-\frac{g}{4}+\frac{D^2}{4I}\left(F_R-F_L\right)$. When this system is restated in the form of a matrix, it

becomes

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}=\begin{bmatrix}\frac{1}{m}-\frac{D^2}{4I} & \frac{1}{m}+\frac{D^2}{4I}\\ \frac{1}{m}+\frac{D^2}{4I} & \frac{1}{m}-\frac{D^2}{4I}\end{bmatrix}\begin{bmatrix}F_R\\ F_L\end{bmatrix}-\frac{g}{4}\begin{bmatrix}1\\ 1\end{bmatrix} \tag{1}$$

To simplify this matrix, let $a=\frac{1}{m}-\frac{D^2}{4I}$, $b=\frac{1}{m}+\frac{D^2}{4I}$, and $\vec{G}=\frac{-1}{4}\begin{bmatrix}1\\ 1\end{bmatrix}=h\begin{bmatrix}1\\ 1\end{bmatrix}$ where $h=\frac{-1}{4}g$

This can be further simplified by designating $\vec{A}=\begin{bmatrix}a & b\\ b & a\end{bmatrix}$ and $\vec{F}=\begin{bmatrix}F_R\\ F_L\end{bmatrix}$. These simplified

expressions are substituted into the matrix equation to give $\ddot{\vec{X}}=A\vec{F}+\vec{G}$. It is important to note that the spring-damper unit is not taken as a point-sized unit and therefore has different forces acting at different points along its vertical length. The sum of the forces acting on these units can be written in terms of each of their end points, x_1 and x_3 for the left side and x_2 and x_4 for the right. The sum of the forces for the left and right spring-damper units are given as $F_L = (x_3 - x_1)k_L + \left(\dot{x}_3 - \dot{x}_1\right)c_L$ and $F_R = (x_4 - x_2)k_R + \left(\dot{x}_4 - \dot{x}_2\right)c_R$, respectively. Utilizing the same steps as those that were used to solve for $\ddot{x}_1$ and $\ddot{x}_2$ lets these two equationsbe written into matrix form to become

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} a(-x_2k_R - v_2c_R) + b(-x_1k_L - v_1c_L) \\ b(-x_2k_R - v_2c_R) + a(-x_1k_L - v_1c_L) \end{bmatrix} + \begin{bmatrix} a\left(x_4k_R + \dot{x}_4c_R\right) + b\left(x_3k_L + \dot{x}_3c_L\right) \\ b\left(x_4k_R + \dot{x}_4c_R\right) + a\left(x_3k_L + \dot{x}_3c_L\right) \end{bmatrix} \tag{2}$$

When simplified using the same expressions as those used in the previous matrix, the first term becomes $\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} F_R \\ F_L \end{bmatrix}$. The second termin this sum is represented under a single vector for simplicity: $\vec{B} = \begin{bmatrix} a\left(x_4k_R + \dot{x}_4c_R\right) + b\left(x_3k_L + \dot{x}_3c_L\right) \\ b\left(x_4k_R + \dot{x}_4c_R\right) + a\left(x_3k_L + \dot{x}_3c_L\right) \end{bmatrix}$. The two matrices can be combined into one simplified matrix as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -bk_L & -ak_R & -bc_L & -ac_R \\ -ak_L & -bk_R & -ac_L & -bc_R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h \\ h \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \overrightarrow{B_1} \\ \overrightarrow{B_2} \end{bmatrix}.$$

To further simplify this system, it is assumed that (X3=)…

$$\vec{B} = \begin{bmatrix} (ak_R + bk_L)A_0\sin(\omega t) + (A_0\omega)(ac_R + bc_L)\cos(\omega t) \\ (bk_R + ak_L)A_0\sin(\omega t) + (A_0\omega)(bc_R + ac_L)\cos(\omega t) \end{bmatrix}$$

Let $k_R = k_L = k$ and $c_R = c_L = c$

$$\vec{B} = \begin{bmatrix} (a+b)kA_0 \sin(\omega t) + A_0 \omega c (a+b)\cos(\omega t) \\ (a+b)kA_0 \sin(\omega t) + A_0 \omega c (a+b)\cos(\omega t) \end{bmatrix}$$

$$a+b = \left[\frac{1}{m} - \frac{D^2}{4I}\right] + \left[\frac{1}{m} + \frac{D^2}{4I}\right] = \frac{2}{m}$$

$$\vec{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(\frac{2kA_0}{m}\sin(\omega t) + \frac{2cA_0\omega}{m}\cos(\omega t)\right) = \frac{2A_0}{m}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(k\sin(\omega t) + c\omega\cos(\omega t)\right)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -bk & -ak & -bc & -ac \\ -ak & -bk & -ac & -bc \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g' \\ g' \end{bmatrix} + \frac{2A_0}{m}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\left(k\sin(\omega t) + c\omega\cos(\omega t)\right)$$

Running the simulation in MATLAB shows a floor whose right endpoint is anchored to its equilibrium position. The left endpoint in set into motion such that the entirety of the floor appears to pivot on its right endpoint. At any point in time, the angle made between the floor and the horizontal that runs through the right end is less than 45 degrees from that horizontal, and that angle gradually decreases as a result of the damper as the floor moves. Eventually, the floor comes to rest at its equilibrium position, where it is perfectly horizontal.

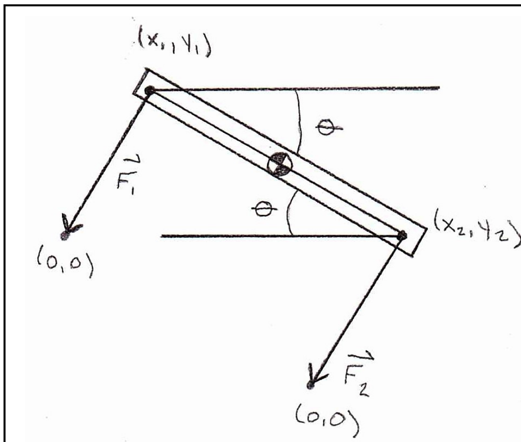

Figure 11: General force diagram for Model Two

III. Three-Degree of Freedom (Model 2)

The second model is a system with three degrees of freedom, allowing for motion in the vertical and horizontal directions along with rotation about the center of mass of the floor. Motion in the downward and leftward directions and counterclockwise rotation are considered to be positive.This model also includes an outside force that pushes on the floor from above. Examples of this might be general human activity and sudden changes of force from above floors due to their movement. Figure 11 shows that the sum of the forces acting on the floor is $\sum\left(\overrightarrow{F_1}+\overrightarrow{F_2}\right)=m\vec{a}$ and the sum of the torques is

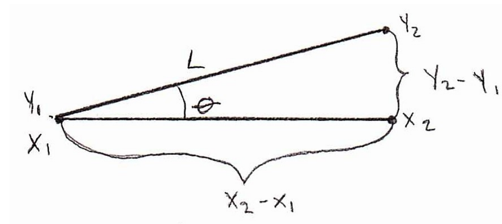

Figure 12: Length-based relationships in Model Two

$\sum\vec{\tau}=\left(\frac{-L}{2}\right)mg\cos\theta+\left(\frac{L}{2}\right)\langle\cos\theta,\sin\theta\rangle\times\overrightarrow{F_2}$. As Figure 12 shows, the length of the floor can berepresented as $L=\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2$. If this is set equal to 0 and manipulated with algebra, the statement becomes $0=\frac{\left(y_2-y_1\right)}{\left(x_2-x_1\right)}\frac{\left(\dot{y}_2-\dot{y}_1\right)}{\left(\dot{x}_2-\dot{x}_1\right)}$.Also based on Figure 12, $\tan\theta=\frac{\left(y_1-y_2\right)}{\left(x_2-x_1\right)}$. This expression and its derivative, $\tan\dot{\theta}=\frac{\left(\dot{y}_1-\dot{y}_2\right)}{\left(\dot{x}_2-\dot{x}_1\right)}$, yield $1=-\tan\theta\tan\dot{\theta}$ after being substituted into the previous equation.

In this model, like the last, the spring and the damper are considered to be in and acting on the same points on the floor and the ground, those points being the $(0,0)$ points on the ground, $\left(x_1,x_2\right)$, and $\left(y_1,y_2\right)$. This allows the force acting on each contact point between the floor and the spring/damper unit to be based on and written as a single point. As such, both of these force equations are based on the general model for vibrational analysis and are given as

$\overrightarrow{F_1} = -k\langle X_1, Y_1\rangle - c\left\langle \dot{X}_1, \dot{Y}_1\right\rangle$ and $\overrightarrow{F_2} = -k\langle X_2, Y_2\rangle - c\left\langle \dot{X}_2, \dot{Y}_2\right\rangle$. When substituted into the equation for the sum of forces, these force expressions yield $m\vec{a} = -k\left[\langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle\right] - c\left[\left\langle \dot{X}_1, \dot{Y}_1\right\rangle - \left\langle \dot{X}_2, \dot{Y}_2\right\rangle\right]$. Dividing this equation by the mass of the floor in order to isolate its acceleration gives $\overrightarrow{a_{cm}} = \frac{-k}{m}\langle X_1 + Y_1, X_2 + Y_2\rangle - \frac{c}{m}\left\langle \dot{X}_1 + \dot{Y}_1, \dot{X}_2 + \dot{Y}_2\right\rangle$. The equation can be further simplified through an elimination of the sums of coordinate points. This is done by stating the coordinate points of the center of mass rather than those of floor's end points. The coordinate points of the center of mass are written in terms of the previously given coordinate points because those are known values with known relationships.The coordinates of the center of mass are then given by $X_{cm} = \frac{X_1 + X_2}{2}$ and $Y_{cm} = \frac{Y_1 + Y_2}{2}$. With these coordinates, simplified equation becomes $\left\langle \ddot{X}_{cm}, \ddot{Y}_{cm}\right\rangle = \overrightarrow{a_{cm}} = \frac{-2k}{m}\langle X_{cm}, Y_{cm}\rangle - \frac{2c}{m}\left\langle \dot{X}_{cm}, \dot{Y}_{cm}\right\rangle$.

It's worth noting that the equation above is a combined equation for $\ddot{X}_{cm}$ and $\ddot{Y}_{cm}$, so it counts as two differential equations. These equations can be converted into a system of equations and stated in matrix form. This is done by restating $\dot{X}_{cm} = V_X$ and $\dot{Y}_{cm} = V_Y$ then rewriting the original equations, which are now expressions for $\dot{V}_X$ and $\dot{V}_Y$, in terms of V_X and V_Y. The system is then stated as the multiplication of two matrices:

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{V}_X \\ \dot{V}_Y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2k}{m} & 0 & \frac{-2c}{m} & 0 \\ 0 & \frac{-2k}{m} & 0 & \frac{-2c}{m} \end{bmatrix} \begin{bmatrix} X \\ Y \\ V_X \\ V_Y \end{bmatrix} \tag{3}$$

However, this portion of the system only encompasses forces and motion in the horizontal and vertical directions, not torques and rotation. Previously, the sum of the torques acting on the floor was given by $I\alpha = I\ddot{\theta} = \sum \tau = \frac{L}{2}\langle \cos\theta, \sin\theta \rangle \times \overrightarrow{F_1} + \frac{L}{2}\langle -\cos\theta, -\sin\theta \rangle \times \overrightarrow{F_2}$.

In order to complete this sum, the two forces acting on the floor must be stated in terms of their locations of action on the floor. These are $\overrightarrow{F_1} = -k\langle X_1, Y_1 \rangle - c\left\langle \dot{X}_1, \dot{Y}_1 \right\rangle = \left\langle -kX_1 - c\dot{X}_1, -kY_1 - c\dot{Y}_1 \right\rangle$

and $\overrightarrow{F_2} = -k\langle X_2, Y_2 \rangle - c\left\langle \dot{X}_2, \dot{Y}_2 \right\rangle = \left\langle -kX_2 - c\dot{X}_2, -kY_2 - c\dot{Y}_2 \right\rangle$. Both expressions are substituted into the previous equation and algebraic manipulation is used to relocate the coefficient of the sum to the left side. The cross products are then carried out to give

$$\frac{2I\ddot{\theta}}{L} = \cos\theta\left[-kY_1 - c\dot{Y}_1\right] - \sin\theta\left[-kX_1 - c\dot{X}_1\right] + (-\cos\theta)\left[-kY_2 - c\dot{Y}_2\right] - (-\sin\theta)\left[-kX_2 - c\dot{X}_2\right].$$

The expression is simplified through the combination of like terms:

$\frac{2I\ddot{\theta}}{L} = \cos\theta\left[k(Y_2 - Y_1) + c\left(\dot{Y}_2 - \dot{Y}_1\right)\right] + \sin\theta\left[k(X_1 - X_2) + c\left(\dot{X}_1 - \dot{X}_2\right)\right]$. At this point, the difference between the locations of the contact points are written in terms of other, constant terms for the floor. These are given as $Y_2 - Y_1 = L\sin\theta$, $\dot{Y}_2 - \dot{Y}_1 = L\cos\theta\dot{\theta}$, $X_2 - X_1 = L\cos\theta$, and $\dot{X}_2 - \dot{X}_1 = -L\sin\theta\dot{\theta}$. When these are substituted into the equation, the equation becomes

$\frac{2I\ddot{\theta}}{L} = \cos\theta\left[kL\sin\theta + cL\cos\theta\dot{\theta}\right] + \sin\theta\left[kL\cos\theta - cL\sin\theta\dot{\theta}\right]$, which can be simplified to

$\ddot{\theta} = \frac{kL^2}{2I}(2\sin\theta\cos\theta) + \frac{cL^2}{2I}\left(\cos^2\theta - \sin^2\theta\right)\dot{\theta}$ with the use of factoring and other algebraic manipulation. The second part of this sum can be written in a way that is more manageably solved through the use of the trigonometric identities for cosine and sine, given as

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$. Use of these gives $\ddot{\theta} = \frac{kL^2}{2I}(\sin 2\theta) + \frac{cL^2}{2I}(\cos 2\theta)\dot{\theta}$.

The solution of this equation will involve theta. Like in the first model, small angle approximation will prove that the error present in the solution will be very small, so long as theta is also small. The Taylor series representation for the sine function is $\sin x = x - \frac{1}{3}x^3 + \frac{1}{60}x^5 + \ldots$ and that for the cosine function is $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$; these show that sine can be approximated as theta and cosine can be approximated as 1. This leads to $\ddot{\theta} = \frac{kL^2}{2I}(2\theta) + \frac{cL^2}{2I}\dot{\theta}$.

To add this differential equation to the matrix system, the equation must be written in terms of omega, where $\dot{\theta} = \omega$. When this is done, the original equation becomes $\dot{\omega} = \frac{kL^2}{2I}(2\theta) + \frac{cL^2}{2I}\omega$.

These equations lead to the matrix $\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{kL^2}{I} & \frac{cL^2}{2I} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$, which is combined with the previous matrix for the full system.

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{V}_X \\ \dot{V}_Y \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-2k}{m} & 0 & \frac{-2c}{m} & 0 & 0 & 0 \\ 0 & \frac{-2k}{m} & 0 & \frac{-2c}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{kL^2}{I} & \frac{cL^2}{2I} \end{bmatrix} \begin{bmatrix} X \\ Y \\ V_X \\ V_Y \\ \theta \\ \omega \end{bmatrix} \tag{4}$$

Upon running this simulation, the differences in range of motion between the first simulation and this one are apparent. In this simulation, the floor begins its motion in the downward and leftward directions such that it rebounds from this motion at an angle. As the motion of the floor continues, it is evident that the center of mass of the floor moves between its bottom left and top right corners of its range of motion while the floor itself rotates about its center of mass. Each of these movements decreases in magnitude as a result of the spring-damper units.

Bibliography

[1] Ali, Mir M. "Evolution of Concrete Skyscrapers: from Ingalls to Jin Mao." *Evolution of Concrete Skyscrapers*, 14 Feb. 2001, www.ejse.org/Archives/Fulltext/200101/01/20010101.htm.

[2] "Building Big: Skyscraper Basics." *PBS*, Public Broadcasting Service, 2000, www.pbs.org/wgbh/buildingbig/skyscraper/basics.html.

[3] "Faultline: Seismic Science at the Epicenter." *Exploratorium*, www.exploratorium.edu/faultline/damage/building.html.

[4] G, Padmanabhan. "Earthquake Resistant Design Techniques for Buildings and Structures."*The Constructor*, 6 Nov. 2017, theconstructor.org/earthquake/earthquake-resistant-techniques/5607/.

[5] Khan, Yasmin Sabina. "Fazlur Khan - Structural Artist of Urban Building Forms." *Princeton University*, The Trustees of Princeton University, 2011, khan.princeton.edu/khan.html.

[6] Lago, Alberto, et al. "Damping Technologies for Tall Buildings: New Trends in Comfort and Safety." Elsevier, 2018, www.elsevier.com/physical-sciences/engineering-and-technology/article-selections/damping-technologies-for-tall-buildings-new-trends-in-comfort-and-safety.

[7] "The Mechanism and Applications of a Tuned Mass Damper | BSBG." BSBG | Brewer SmithBrewer *Group*, 22 Nov. 2017, bsbgltd.com/blog/the-mechanism-and-applications-of-tuned-mass-damper-tmd/.

[8] Mirsha, Renu, and A. K. Dwivedi. "Shear Wall and Frame Interaction Terminology." GlobalResearch *and Development Journal for Egineering*, vol. 1, no. 7, June 2016, www.grdjournals.com/uploads/article/GRDJE/V01/I07/0055/GRDJEV01I070055.pdf.

[9] "Sheds - How to Determine the Stronger Shed." *Shedsales.com.au*, 2017, old.shedsales.com.au/shed-compare_strength.aspx.

[10] "Sears Tower (Currently Willis Tower)." *Princeton University*, The Trustees of Princeton University, 2011, khan.princeton.edu/khanSears.html.

[11] "Thread: One Rincon Hill - Earthquake Technology." *Wired New York Forum RSS*, wirednewyork.com/forum/showthread.php?t=18301.

[12] Yirka, Bob. "Japanese Companies Develop Quake Damping Pendulums for Tall Buildings."*Phys.org - News and Articles on Science and Technology*, 2 Aug.

2013, phys.org/news/2013-08-japanese-companies-quake-damping-pendulums.html.