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Guitars: The Sound of Mathematics

I. Background

Our human senses allow us to interact with the environment, and fully immerse ourselves into it. Without any of these senses, one can't experience life as it should be. For some, senses are what give meaning to their lives. However, some favor specific senses more than others. Take, for instance, musicians. For them, the hearing sense is a crucial sense of theirs that may ultimately have led them to pursue a music career and an interest in playing instruments. Perhaps the easiest instruments to explain sound with are string instruments, such as the guitar.

The history of guitars can be traced back to predate written history, with the oud and the lute. The origin of the guitar has close ties with Christianity as it was said that a man named Lamech, who was Noah's grandfather and the sixth grandson of Adam and Eve, designed the Arab precursor to the guitar. After hanging the body of his dead son from a tree, Lamech was able to come up with the shape of the oud instrument. Another instrument called the lute also became prominent as it was passed from the Egyptians to the Greeks and then onto the Romans, who took it to Europe. Although the lute was a four-string instrument, the oud contained five strings. Some believe that the oud was inspired by the lute since oud is described as a "fretless, plucked short-necked lute with a body shaped like half a pear." By the end of the Renaissance, the lute had immensely evolved over time to the point where most lutes had between 20 and 30 strings.

Unfortunately, the shape of the instrument lost popularity and instead a curved shape won over 15th and 16th century Spaniard musicians. This led to the creation of Baroque guitars, which contained five strings and introduced the inclusion of a fretboard. Another instrument like the guitar, called the vihuela, became popular in Spain and was used commonly by Mariachis. By the 1970s, the evolution of Spanish guitars settled as the guitars were smaller and had six strings[1].

MatLab offers many tools to users to create many more things. One of these things includes making a guitar with proper guitar sounds using a synthesizer's technology. A synthesizer is an electronic musical instrument that generates sound via vibrations using electronic devices such as oscillators, modulators, and carriers. The history of the synthesizer can be traced back to a 7-ton machine around the start of the 20th century that used motors to produce electricity and telephone receivers to turn them into sound. This machine was known as the telharmonium and was shut down by former American electronics company The RCA Corporation before they created Mark II, which is often thought of as the first programmable synthesizer. Affectionately named 'Victor', Mark II was made at Columbia University and used a paper tape reader as its sequencer[2]. Later, the synthesizer had many advancements such as the inclusion of a Voltage Controlled Oscillators (VCOs), noise generators, and Frequency Modulation, all of which help make synthesizers sound how they are now.

No matter what sound is produced, each sound takes the shape of a wave. Depending on the sound produced, the soundwave created could vary. For instance, a quieter sound produces sound waves with a small amplitude, whereas a louder sound produces a large amplitude. Similarly, a lower-pitch sound produces sound waves with longer wavelengths, whereas a higher-pitch sound produces shorter wavelengths[3]. It's also important to note that there is not just one soundwave being produced when a string is plucked but instead, a series of soundwaves are produced[4].

II. Mathematics in Instruments

Soundwaves are three-dimensional, each with length, width, and height. However, this makes it more difficult to mathematically represent a wave. Instead, we can represent a wave using a one-dimensional wave equation that focuses on wave propagation along a single axis. One-dimensional soundwaves can be represented by the following one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \tag{1}$$

To specify a given wave, we define it by the following boundary conditions,

$$\psi(0, t) = 0 \tag{2}$$

$$\psi(L, t) = 0, \tag{3}$$

where there are fixed endpoints at

$$x = 0 \tag{4}$$

$$x = L, \tag{5}$$

and the following initial conditions:

$$\psi(x, 0) = f(x) \tag{6}$$

$$\frac{\partial\psi}{\partial t}(x,0) = g(x) \tag{7}$$

This is because when you pluck a guitar string, the string's ends will never change elevation. A visual representation would look something like this:

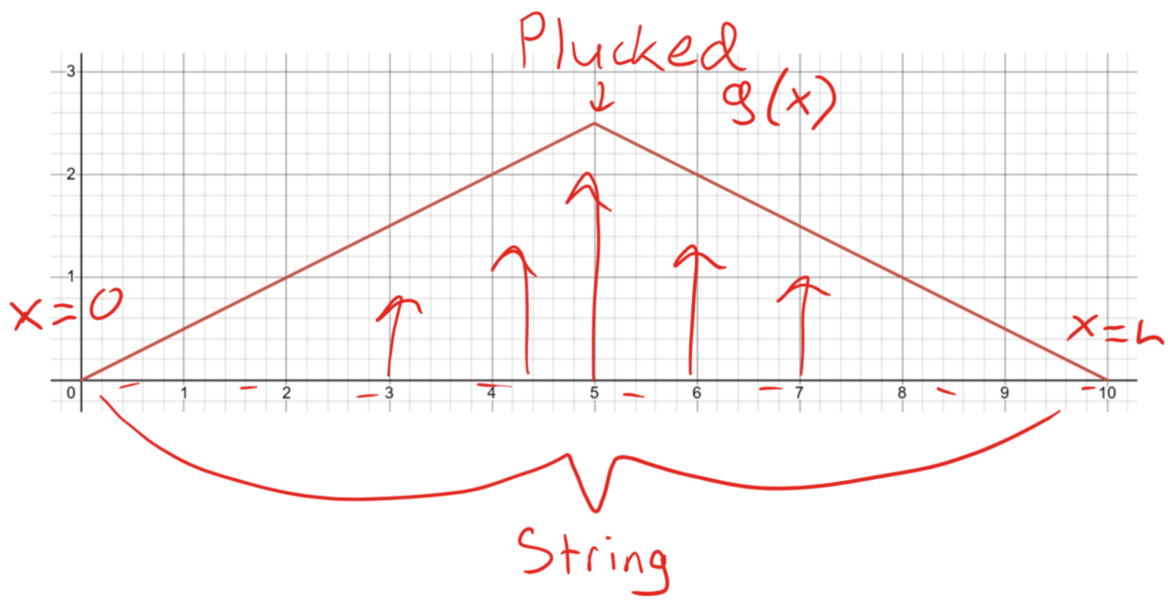

Figure 1: Graphical representation of a plucked string $g(x)$ with endpoints at $x = 0$ and $x = L$. In this representation, the string is plucked at the center point.

There are multiple solutions to this equation, such as the separation of variables solution, which uses a trial solution:

$$\psi(x,t) = X(x)T(t) \tag{8}$$

Where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone. If we take the double derivative of both $X(x)$ and $T(t)$, this leads to

$$T\frac{d^2X}{dx^2} = \frac{1}{v^2}X\frac{d^2T}{dt^2} \tag{9}$$

With this in mind, let's assume the following:

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{v^2}\frac{1}{T}\frac{d^2T}{dt^2} = -k^2 \tag{10}$$

This gives us the solution for X, which is

$$X(x) = Ccos(kx) + Dsin(kx) \tag{11}$$

If we rewrite (10), we can get

$$\frac{1}{T}\frac{d^2T}{dt^2} = -v^2k^2 = -\omega^2 \tag{12}$$

This gives us the solution for T, which is

$$T(t) = Ecos(\omega t) + Fsin(\omega t), \tag{13}$$

where

$$v = \frac{\omega}{k} \tag{14}$$

After setting the boundaries

$$\psi(0, t) = \psi(L, t) = 0, \tag{15}$$

we get the following:

$$C = 0 \tag{16}$$

$$kL = m\pi, \tag{17}$$

where m is an integer. For a particular value of m, we can rewrite the equation as

$$\psi_m(x, t) = [E_m \sin(\omega_m t) + F_m \cos(\omega_m t)]D_m \sin\left(\frac{m\pi x}{L}\right) \tag{18}$$

$$\equiv [A_m \cos(\omega_m t) + B_m \sin(\omega_m t)] \sin\left(\frac{m\pi x}{L}\right) \tag{19}$$

With the following initial condition taken into consideration

$$\psi(x, 0) = 0 \tag{20}$$

this results in

$$B_m = 0, \tag{21}$$

so (19) becomes

$$\psi_m(x, t) = A_m \cos(\omega_m t) \sin\left(\frac{m\pi x}{L}\right) \tag{22}$$

Since the general solution is a sum of all possible values of m, the solution is

$$\psi(x, t) = \sum_{m=1}^{\infty} A_m \cos(\omega_m t) \sin\left(\frac{m\pi x}{L}\right) \tag{23}$$

under the remaining initial conditions listed in (6) and (7)[5].

III. Discretization

There is also a numerical solution to the one-dimensional wave equation, also known as discretization. Finite difference methods can be used to discretize this equation. This involves the Taylor series, which states that

$$u_{n+1} = u(x + \Delta x) = u(x) + u'(x)\Delta x + u''(x)\frac{\Delta x^2}{2} + u'''(x)\frac{\Delta x^3}{6} + \cdots \tag{24}$$

We can write a function $f(x)$ as a series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n, \tag{25}$$

where

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{26}$$

We can rewrite this as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots \tag{27}$$

We can also use α to explain this, where

$$\alpha_{\ 1} = u_{n-1} = u(x_n - \Delta x) = u(x_n) + u'(x_n)\Delta x + \frac{u''(x_n)}{2}\Delta x^2 + \frac{u'''(x_n)}{6}\Delta x^3 + \cdots \tag{28}$$

$$\alpha_2 = u_n = u(x_n) \tag{29}$$

$$\alpha_{\ 3} = u_{n+1} = u(x_n + \Delta x) = u(x_n) - u'(x_n)\Delta x - \frac{u''(x_n)}{2}\Delta x^2 - \frac{u'''(x_n)}{6}\Delta x^3 - \cdots \tag{30}$$

So, we can write the equation as

$$u'(x) \approx \alpha_{\ 1} u_{n-1} + \alpha_{\ 2} u_n + \alpha_{\ 3} u_{n+1} \tag{31}$$

Final

IV. MatLab and Instrument Design

Pitch is important in instruments like the guitar. Fortunately, the guitar's mechanics are perfect for tuning. The guitar's strings are labeled chronologically E, A, D, G, B, and E. The first

E plays the lowest-pitch note while the second E plays the highest-pitch note. In order for a guitar to function properly, the strings must play the correct notes[6].

We can use MatLab to simulate a plucked string. Consider a guitar string, where both ends are fixed at a height of 0. So the update rules for future(1) and future(n) can set the values to 0. We can simulate a plucked string with the following:

```
dx = .01;          % Spacing of points on string
dt = .001;         % Size of time step

c = 5;             % Speed of wave propagation
L = 10;            % Length of string
stopTime = 30;     % Time to run the simulation

r = c*dt/dx;
n = L/dx + 1;

% Set current and past to the graph of a plucked string
current = .5-.5*cos(2*pi/L*[0:dx:L]);
past = current;

for t=0:dt:stopTime
  % Calculate the future position of the string
  future(1) = 0;
  future(2:n-1) = r^2*(current(1:n-2)+current(3:n)) + 2*(1-r^2)*current(2:n-1) - past(2:n-1);
  future(n) = 0;

  % Set things up for the next time step
  past = current;
  current = future;

  % Plot the graph after every 10th frame
  if mod(t/dt, 10) == 0
    plot([0:dx:L], current)
    axis([0 L -2 2])
    pause(.001)
  end
end
```

Figure 2: Code snippet for graphically and mathematically describing the one-dimensional wave equation, utilizing the Taylor series to calculate the future position of the string[7].

In this snippet, the future time is set up so that it follows the discretization solution of the wave equation. However, instead of points being written as $n-1$, n, and $n+1$, it is written as $n-2$, $n-1$, and n. We know that

$$u_{tt} = c^2 u_{xx}, \tag{32}$$

and that

$$u_{xx} = \alpha_1 u_{n-1} + \ \alpha_2 u_n + \alpha_3 u_{n+1} \tag{33}$$

We can rewrite this as

$$u_{xx} = \frac{u_{n-1} - 2u_n + u_{n+1}}{(\Delta x)^2}, \tag{34}$$

but if we want to include the constant c, or in this case r, then this leads to $r^2 \frac{(u_{j-1}^n - 2u_j^n + u_{j+1}^n)}{(\Delta x)^2}$.

After canceling terms via substitution, we can isolate u_j^{n+1}, where

$$u^{n+1} = r^2(u_{j-1}^n - 2u_j^n + u_{j+1}^n) \tag{35}$$

$$r = \frac{\Delta t}{\Delta x} \tag{36}$$

This leads to our final result:

$$u_j^{n+1} = r^2\left(u_{j-1}^n + u_{j+1}^n\right) + 2u_j^n(1 - r^2) - u^{n-1} \tag{37}$$

Final

MatLab also includes a built-in tool that can simulate the actual sound of a plucked string, which follows a similar layout to the code above.

```
% Parameters
L = 0.65; % Length of the string (meters)
T = 100;  % Tension (Newtons)
rho = 0.01; % Linear density (kg/m)
v = sqrt(T/rho); % Wave velocity
Fs = 44100; % Sampling frequency (Hz)
duration = 2; % Duration of the sound (seconds)
n_samples = duration * Fs;

% Discretize the string
dx = L / 100; % Spatial step
dt = 1 / Fs;  % Time step
x = 0:dx:L;   % String positions
n_points = length(x);

% Initial condition: pluck the string at the center
y = zeros(n_samples, n_points); % Displacement matrix
pluck_position = round(n_points / 2);
y(1, :) = sin(pi * (1:n_points) / n_points); % Example initial displacement

% Time-stepping loop for finite difference method
for t = 2:n_samples-1
    y(t+1,2:end-1) = 2 * y(t,2:end-1) - y(t-1,2:end-1) + ...
        (v*dt/dx)^2 * (y(t,3:end) - 2*y(t,2:end-1) + y(t,1:end-2));
end

% Extract signal at one point (e.g., near the bridge)
signal = y(:, end);

% Play the sound
sound(signal, Fs);
```

Figure 3: An alternative version of the previous code snippet that plays a sound in MatLab when the program is run to mimic the sound of a plucked string given the required parameters and initial conditions.

The code shown above can produce graphs where the length of the string is 0.65 meters with the sound playing for two seconds, which gives us this graph:

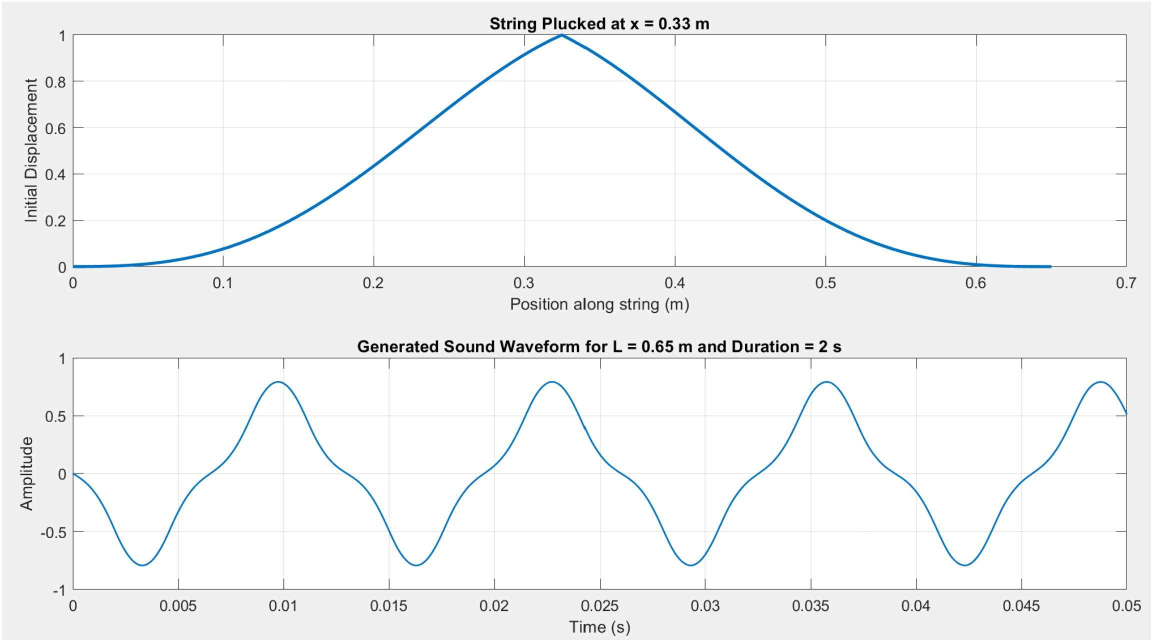

Figure 4: Recorded graph of a 0.65-meter string plucked at x = 0.33 meters with a duration of 2 seconds. The top graph shows the string length and the location of the string being plucked. The bottom graph shows a snippet of the generated sound waveform for a small amount of time.

However, we can turn this into a function that takes in user inputs for string length and duration as well as where the string is plucked to get other graphs, such as these:

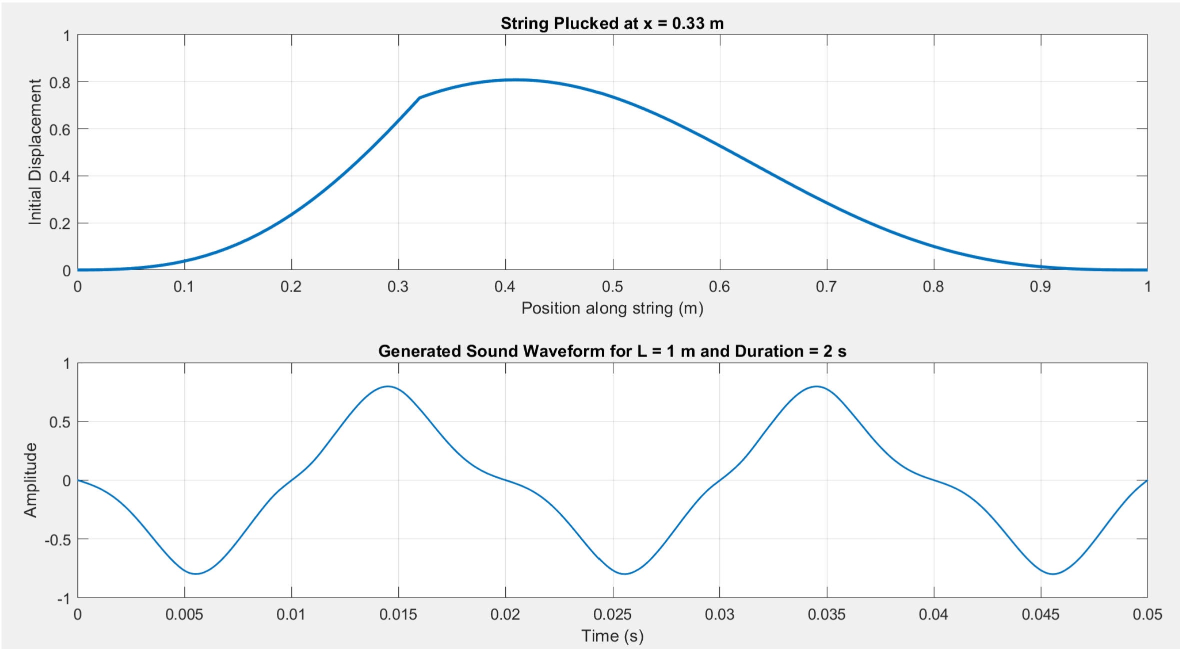

Figure 5: Recorded graph of a 1-meter string plucked at x = 0.33 meters with a duration of 2 seconds. The top graph shows the string length and the location of the string being plucked. The bottom graph shows a snippet of the generated sound waveform for a small amount of time.

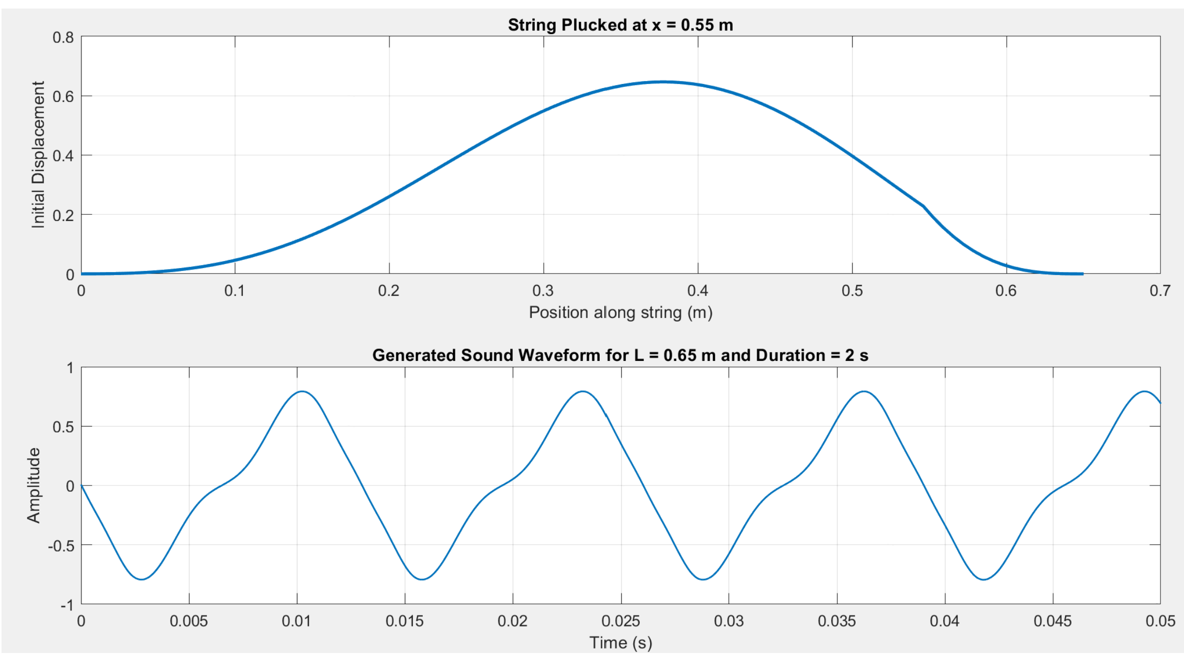

Figure 6: Recorded graph of a 0.65-meter string plucked at x = 0.55 meters with a duration of 2 seconds. The top graph shows the string length and the location of the string being plucked. The bottom graph shows a snippet of the generated sound waveform for a small amount of time.

V. Error Analysis

It's important to note that the one-dimensional wave equation is prone to compounding errors. Let's consider the following:

$$\frac{\partial^2 u}{2t^2} = k^2 \frac{\partial^2 u}{\partial x^2} \tag{38}$$

With this, we can also consider the following approximation:

$$\frac{u_j^{n-1} - 2u_j^n + u_j^{n+1}}{(\Delta t)^2} \approx k^2 \left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2} \right) \tag{39}$$

After canceling the deltas, and changing the k to r to fit the context, we get

$$u_j^{n-1} - 2u_j^n + u_j^{n+1} \approx r^2 \left(u_{j-1}^n - 2u_j^n + u_{j+1}^n \right) \tag{40}$$

To find the error, we can isolate the u_j^{n+1} to one side of the equation.

$$u_j^{n-1} \approx r^2 \left(u_{j-1}^n - 2u_j^n + u_{j+1}^n \right) - u_j^{n+1} + 2u_j^n \tag{41}$$

This gives us

$$u\left(x_j, t_{n+1}\right) = r^2 (\Delta x^2 u'')_j^n + (\frac{\Delta x^4}{12} u^{(4)})_j^n \tag{42}$$

When we include the second-order derivative, we get the error e:

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} - u_{xx}\left(x_j, t_n\right) = e \tag{43}$$

This leads to

$$\frac{(u_j - \Delta x u_j' + \frac{\Delta x^2}{2} u_j'' - \frac{\Delta x^3}{6} u_j''' + \cdots) - 2u_j + (u_j + \Delta x u_j' + \frac{\Delta x^2}{2} u_j'' + \frac{\Delta x^3}{6} u_j''' + \cdots) - u_{xx}}{(\Delta x)^2} = e (44)$$

We can simplify this to the following:

$$\frac{\frac{\Delta x^4 u_j^{(4)}}{4!} + \frac{\Delta x^4 u_j^{(4)}}{4!}}{(\Delta x)^2} = e \tag{45}$$

Where

$$e = (\Delta x^2) u_j^{(4)} (\frac{1}{12}) \tag{46}$$

This gives us

$$e \sim O(\Delta x^2) \tag{47}$$

Final

VI. Conclusion

There's more mathematics involved in sound than one may think. Although a soundwave could be seen as a sine wave on the surface, they can also be represented as calculus equations with first and second-order derivatives and Taylor series. MatLab serves as an incredible tool to visualize this and even simulate sounds to resemble a plucked string. Perhaps in the future, many other instruments can be simulated like this.

Works Cited

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