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Modern Differential Equations

I. Abstract

With the recent success of entrepreneurial actors in the space industry, renewed interest in space exploration is ubiquitous. Unsurprisingly, the math involved with these technological feats is heavily dependent on differential equations and numerical solutions. The focus of this paper is to present the historical evolution of rocket propulsion technology, and to use differential equations, numerical methods, and MATLAB programming to model the 1969 Apollo 11 mission that first put men on the Moon. The model relies on data provided by NASA to model the geocentric orbit and trans-lunar trajectory. The results of the modeling are shown to be consistent with the figures in the mission description provided by NASA.

II. History of Rocketry

The first true rockets emerged in thirteenth century China as an innovation in warfare. Chinese soldiers fighting invading Mongol armies used a crude version of what would be classified today as a solid propellant rocket. By filling hollow tubes with gunpowder and leaving one end open, the Chinese discovered that the fire, smoke, and gas emitted from the rocket would propel arrows to greater velocities and to reach greater ranges. The "rocket" could then be attached to an arrow or a long stick for increased accuracy. Similar methods were used for the launching of fireworks. Throughout the next few centuries experiments with rocketry continued with increasing sophistication. Around the same time period as the Chinese started using gunpowder propelled rockets, the monk Roger Bacon of England worked on creating improved forms of gunpowder, which allowed for rockets to have an increased range. About a century later in France, Jean Froissart found that by launching rockets through tubes the accuracy of the flight could be greatly improved. Johannes Fontana of Italy used rocketry technology to create a torpedo for setting enemy warships on fire. In the sixteenth century, the German fireworks maker Johann Schmidlap invented the first multistage rocket to propel fireworks to greater altitudes. While the mathematical framework for modern rocket science was laid out by Isaac Newton and other mathematicians in the 1600s, rocketry was used exclusively for warfare throughout the eighteenth and nineteenth centuries. It would not be considered as a means of space travel until the turn of the 20th century.

Astronautics developed separately in the U.S, Russia, and Europe. Consequently, the title "father of astronautics" is attributed to three different men. These are Konstantin Tsiolkovsky, Robert H. Goddard, and Hermann Oberth who were Russian, American, and Romanian respectively. Their shared ownership of the title is attributed to the lack of evidence that these individuals were aware of each other's work.

Tsiolkovsky first worked with aeronautics by creating the first Russian wind tunnel in 1897. He used the tunnel to test the aerodynamics of different aircraft designs, specifically studying the effects of air friction and surface area on the air speed over the aircraft body. He later moved on to consider space travel, publishing perhaps his most significant contribution to astronautics in 1903, the rocket equation. The equation is based on the conservation of momentum and describes the relationship between the increase in speed of the rocket, the effective exhaust velocity, and the mass of the rocket and propellant burned. The equation is represented,

$$\Delta v = v_e \ln\left(\frac{m_0}{m_f}\right) \qquad \text{Eq. 1}$$

where the change in the velocity of the rocket is represented as Δv, the effective exhaust velocity as v_e, the original mass of the rocket including propellant as m_0, and the final mass of the rocket after all propellant has been burned as m_f. Like the other two fathers of astronautics, Tsiolkovsky was especially interested in liquid propellant engines and multi-stage rockets. He published his theory of multi-stage rockets in 1929 where he calculated the velocity required to escape Earth's gravity to be 8 km/s. He argued that a multi-stage rocket with a liquid engine that used liquid oxygen and hydrogen as propellant would be able to achieve this velocity. Although Tsiolkovsky never physically constructed a rocket, his rocket equation and work on escape velocities and liquid propellant engines formed the foundation of modern astronautics.

Unlike Tsiolkovsky, the American scientist Robert Goddard did actually experiment with physical rockets. In 1926, in an event which is frequently compared to the first airplane flight, Goddard launched the world's first liquid propellant rocket in Auburn, Massachusetts. The rocket was in flight for 2.5 seconds and achieved a speed of 60 miles per hour. The rocket reached an altitude of 41 feet and landed 184 feet away from the launch site. The rocket was 10 feet tall and fueled by liquid oxygen and gasoline. With respect to theoretical contribution, Goddard was the first to prove that rockets would work in the vacuum of space and that they didn't need air to push off of. He also furthered the development of sophisticated rockets by developing gyro-control and parachute recovery. Goddard's research was not of much interest to the U.S. government at the time, however the Germans, having realized the military applications, based their rocketry programs on his work. Goddard laid the foundational engineering of what later would become the V-2 rocket in World War II. This foundation included gyroscopic control, vanes in the jet stream of the motor for steering, and gimbal-steering. Having made such innovative technological advancements, Goddard set the stage for the rapid development of rocket science during and after the second world war.

Around the same time as Goddard was launching the first liquid propellant rockets, the Romanian-German scientist Herman Oberth was making similar advancements in liquid propellant rocketry in pre-Nazi Germany. Despite his persistence in advocating for the rockets, his work was at first rejected by the government and academia. The War ministry regarded his proposal for a long-range liquid propellant rocket as a fantasy and his 1922 dissertation on rocket design was rejected. Refusing defeat, in 1923 he published the rejected dissertation in his book "The Rocket into Interplanetary Space." The book explained mathematically how to escape the gravitational pull of the Earth with rockets and brought him much recognition. In 1931 he built Germany's first liquid propellant rocket and launched it that same year near Berlin. With respect to theoretical rocket science, the Oberth maneuver is perhaps his most significant contribution. The theoretical spacecraft maneuver is based on the physics that changes in velocity at high velocities produce greater changes in kinetic energy than at lower ones. The theory states that the engine efficiency is greatest when the burn is initiated at the periapsis of an elliptical orbit, where the orbital speed is the greatest. By generating public and government interest in rocketry, Oberth inspired other German scientists to contribute to the development of astronautics. The work initiated by these scientists led directly to the formation of the German space program under the Nazis, and later to the realization of the dream of space travel.

With the rise of the Nazis in Germany, civilian research on rocketry was outlawed. The scientists involved were faced with the tough decision of choosing between abandoning their research or working for the military. One of these scientists that had grown interested in the military applications of rocket research and had earlier worked for Oberth, was Werner von

Braun. In 1932 Von Braun was appointed technical director of the Nazi research program. In 1937, the research group set up camp in the German town of Peenemunde, where they concentrated their efforts on the development of the V-2 rocket. The V-2 was revolutionary technology for the time with its use of gyroscopic guidance and a computer to track its trajectory and correct deviations. It used a liquid propellant engine that ran off liquid ethanol and oxygen and produced 60,000 pounds of thrust which allowed for a maximum velocity of 5,670 kilometers per hour. Each rocket was about 46 feet in height and carried a ton of explosives. It was first used by Germany in 1944, where it targeted allied forces London, Paris, and Belgium. By the end of the war, 4,000 V-2 missiles had been launched. There were no counter measures for the new weapon, but it was developed too late in the war to change the outcome.

As the war drew to a close and the allied forces began to overtake V-2 missile facilities both the Americans and the Soviets were desperate to get their hands on the technology. The U.S. launched operation Paperclip, where Nazi scientists were transported to Fort Bliss, Texas and given American citizenship to work for the U.S. on rocketry. The operation was launched after the Osenberg List, a listing of important Nazi scientists, was found in an improperly flushed toilet. Prominent scientists included Wernher von Braun, Erich W. Neubert, Theodor A. Poppel, August Schulze, and Eberhard F.M Rees. In August of 1945, 127 scientists accepted contracts with the Research and Development Division of Army Ordinance. For many of these scientists, including Von Braun, working for the Americans seemed a much superior option than the communist Soviet Union. Deteriorating relations between the U.S and the Soviet Union led to a race to develop intercontinental ballistic missiles capable of carrying nuclear warheads. In 1950, the team led by Von Braun was transferred to Huntsville, Alabama to work on the Redstone and Jupiter missiles. These V-2 derivatives took the first Americans into space.

In 1960, president Eisenhower established NASA, and appointed Wernher von Braun as chief architect of the Saturn V, the largest rocket ever made. Following the establishment of NASA, president John F. Kennedy in 1961 delivered the famous speech "We Choose to Go to the Moon," where he claimed that the U.S would have men on the Moon by the end of the decade. Kennedy argued that because the exploration of space was an inevitability, the United States should lead the way as modern pioneers. The work of inspiration greatly increased public support of the Saturn V project and secured NASA government funding. Later that year the Apollo program was established.

Leading up to the Apollo 11 mission were several preparatory missions that demonstrated the capabilities of the Saturn V rocket and the command, service, and lunar modules of the spacecraft. These missions began in 1967 and concluded before Apollo 11 in 1969. Before 1967 NASA conducted tests mainly on the thrust capabilities of the first and second stage engines. The later Apollo missions first demonstrated the capabilities of Saturn V and the Apollo spacecraft in space near Earth, building up to control systems tests above the lunar surface. The mission specifics are given in Table 1.

Table 1. Listing of Apollo Missions before Apollo 11

Mission Number	Objectives Completed
Apollo 1	• Failed first manned mission of Apollo. • 3 astronauts died in a pre-flight procedure when a fire spread through the command module
Apollo 4	• Unmanned test flight of the Saturn V • Tested Command Module Heat shield at Re-entry speeds • Demonstrated the third stage restart
Apollo 5	• First test flight of the Lunar Module

	• Successfully fired the ascent and descent engines
Apollo 6	• Second Flight of Saturn V • Service Module is used to achieve re-entry speeds • Saturn V is declared man-rated
Apollo 7	• Demonstrate rendezvous capabilities of Apollo Command and Service Modules • Demonstrate Broadcasting from Space
Apollo 8	• Demonstrate Trans-Lunar Injection • Demonstrate Navigation of Apollo spacecraft and trajectory correction
Apollo 9	• Test capabilities of the Lunar Module • Demonstrate docking of the Lunar Module with the Command Module • Demonstrate Self-Sufficiency of Apollo Spacecraft
Apollo 10	• Operate Spacecraft around the Moon • Orbit the Moon with the Lunar Module • Dock Lunar Module with Command Module in lunar orbit

The Apollo 11 mission finally put men on the Moon using the Saturn V. In addition to the main three stages of propulsion, the rocket contained a lunar module for descent to the lunar surface, a command module to carry the astronauts around the Moon in lunar orbit, and a service module to make corrections in trajectory throughout the mission. The mission used a free return trajectory as a default, where the gravitational pull of the Moon would send the spacecraft back to the Earth if there were no expenditure of propellant. Had there been any in-flight problems on the way to the Moon, the spacecraft would still have returned to Earth. Because a free trajectory to the Moon would not put the Apollo spacecraft close enough to the Moon for a lunar landing, a series of trajectory corrections and orbital insertions were performed by the spacecraft.

The Saturn V stood at 363 feet tall. It consisted of three separate stages, which used the newly developed F-1 and J-2 engines for thrust. The first stage was almost entirely fuel and used liquid oxygen and RP-1 fuel, a highly refined form of kerosene, as propellant. It was powered by five F-1 engines together producing 7.5 million pounds of thrust to accelerate the rocket through the lower atmosphere. The second stage consisted of five J-2 rocket engines and used liquid oxygen and hydrogen. It accelerated the rocket through the upper atmosphere with about 1.1 million pounds of thrust. The third stage used a single J-2 engine and ran off the same fuel as the second stage. It was used twice during the lunar mission, once to push the spacecraft to low Earth orbit and once for trans-lunar insertion. The Saturn V had its first unmanned test flight in 1967 and was last launched in 1973. The details of the Apollo 11 mission are given in Table 2.

Table 2. Apollo 11 Mission Overview

Time of Maneuver in hours after launch	Description of Maneuver
0 – 2:44	Apollo spacecraft and third stage are put into Earth orbit of 114 by 116 miles
2:44	Third stage burns for 5 minutes 48 seconds to put Apollo 11 into trans-lunar trajectory. The Command and Service Modules disconnect from the third stage and Lunar module (LM), flip around, and redock with the Lunar Module.
4:40	Third stage detached from the Apollo spacecraft and goes into heliocentric orbit
75:50	Apollo 11 arrives behind the Moon. Service Propulsion Engine (SPS) puts the spacecraft into an orbit of 69 by 190 miles with a burn time of 357.5 seconds. A second burn time of 17 seconds puts the spacecraft into a nearly circular orbit of 62 by 70.5 miles.
100:12	LM detaches from Command Module.

101:36	LM is Behind the Moon on the 13th orbit. LM fires engines for 30 seconds to commence descent orbit insertion to an orbit of 9 by 67 miles.
102:33	LM commences descent initiation by firing engines for 756.3 seconds. LM is 26,000 feet above lunar surface and 5 miles downrange from landing site.
102:45	Braking thrust is halted, LM lands in the Sea of Tranquility at 0 degrees, 41 minutes, 15 seconds north latitude and 23 degrees, 26 minutes east longitude, 4 miles downrange from expected landing spot.
109:42	Armstrong steps on the Moon
111:39	Astronauts re-enter the LM.
124:22	LM ascent stage fires. Burn time of 435 seconds putting the LM in an orbit of 11 by 55 miles. Command Module is in 25th revolution.
125:19	LM reaction control system puts the LM into a circular orbit of radius 56 miles.
128:03	LM docks with the Command Module.
132:00	LM is jettisoned and remains in lunar orbit
134:00	Trans-Earth injection begins with an SPS burn time of 150 seconds
150:30	Course correction with an SPS burn time of 11.2 seconds
178:00	Re-entry procedures initiated
195:13	Parachute deployment
195:18	Apollo 11 lands in the Pacific Ocean, at 13 degrees, 19 minutes north latitude and 169 degrees, nine minutes west longitude

III. Modeling of the Apollo 11 Lunar Mission

This section of the report will use Tsiolkovsky's rocket equation (Eq.1) and basic orbital mechanics to model the transition from Earth orbit to trans-lunar injection. For simplicity the modeling will treat the Earth-Moon system as a static single body system, the average distance between the Earth and the Moon will be used, and trans-lunar injection will be approximated with an ellipse with the Earth at one focus and the Moon at the other. The calculations will be compared with the numbers reported by NASA, and the error for each calculation will be given. The model in the body of the report will include the necessary equations and results only. The computations are in Appendix A. The required input parameters are displayed in Table 3 and a visual representation of the trajectory is shown in Figure 1.

Table 3. Input Parameters

Input	Value	Symbolic Representation
Gravitational Parameter of Earth	$3.98\times10^{14}\,\frac{m^3}{s^2}$	μ_E
Gravitational Parameter of the Moon	$4.9\times10^{12}\,\frac{m^3}{s^2}$	μ_m
Radius of the Earth	$6.371\times10^{6}\,m$	r_E
Radius of the Moon	$1.737\times10^{6}\,m$	r_m
Average Distance to the Moon	$3.844\times10^{8}\,m$	d
Altitude of Earth Orbit	$185075m$	$r_{E,alt}$
Altitude of Lunar Orbit	$106619m$	$r_{m,alt}$

Specific Impulse of J-2 Engine	$421s$	I_{sp}
Vacuum Thrust of J-2 Engine	$1.0331\times10^{6}N$	F_{thrust}
Total Mass of Saturn V	$2.97\times10^{6}kg$	m_{Total}
Gross Mass of Stage One	$2.29\times10^{6}kg$	m_1
Gross Mass of Stage Two	$496200kg$	m_2
Gross Mass of Stage Three	$123000kg$	m_3
Empty Mass of Stage Three	$13500kg$	$m_{3,e}$
Mass of Launch Escape	$3631kg$	m_{se}
Gravitational Acceleration at the Earth Surface	$9.8m/s$	g_0
First Third Stage Burn Duration	$165s$	t_1
Total Third Stage Burn Duration	$500s$	t_{total}

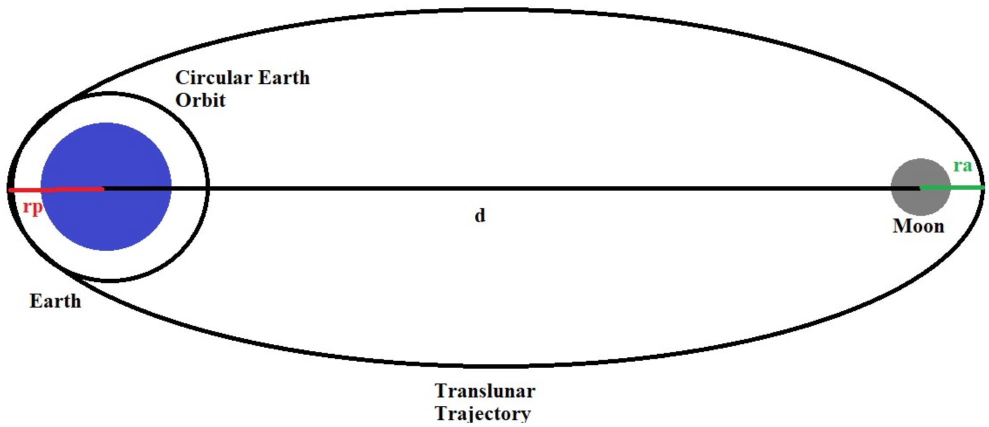

Figure 1. Visual of Trans-lunar Injection

Before any orbital calculations can be made, the values of r_p and r_a must be calculated. As shown in Figure 1, r_p is calculated from the radius of the Earth and the altitude of the Earth orbit, and r_a is calculated from the radius of the Moon and the altitude of lunar orbit. The values are calculated as follows:

$$r_p = r_E + r_{E,alt} \qquad \text{Eq. 2}$$

$$r_a = r_m + r_{m,alt} \qquad \text{Eq. 3}$$

Using the value of r_p and orbital mechanics for circular orbits, the orbital velocity of the Apollo 11 spacecraft can be calculated. The gravitational parameter of Earth is also needed. The orbital velocity (v_0) is calculated from Newton's Second Law (Eq. 4):

$$\frac{\mu_E m}{r^2} = \frac{mv^2}{r} \qquad \text{Eq. 4}$$

Plugging in known values and simplifying the expression reduces Equation 4 to Equation 5.

$$v_0 = \sqrt{\frac{\mu_E}{r_p}} \qquad \text{Eq. 5}$$

The trans-lunar injection trajectory can be approximated by an ellipse where the major axis a, spans half the distance of r_a, r_p, and d. Equation 6 describes the relationship between the major axis a, and the values r_a, r_p, and d.

$$2a = r_p + r_a + d \qquad \text{Eq. 6}$$

The relationship between the velocity of the spacecraft and the elliptical orbit is expressed as Equation 7.

$$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a} \qquad \text{Eq. 7}$$

Plugging in known values and simplifying the expression gives Equation 8. The trans-lunar injection velocity is represented as v_f.

$$v_f = \sqrt{2\left(\frac{\mu_E}{r_p} - \frac{\mu_E}{2a}\right)} \qquad \text{Eq. 8}$$

The Δv for the maneuver is the difference of v_f and v_0.

$$\Delta v = v_f - v_0 \qquad \text{Eq. 9}$$

Equation 9 yields a Δv of 3,129 meters per second. Alternatively, the Δv can be calculated with Tsiolkovsky's rocket equation (Eq. 1).

$$\Delta v = v_e \ln\left(\frac{m_0}{m_f}\right) \qquad \text{Eq. 1}$$

Equation 1 requires the mass of the spacecraft before and after the burn. The mass flow rate and effective exhaust velocity of the engine will need to be calculated given the values of the specific impulse and vacuum thrust. Equation 10 calculates the effective exhaust velocity (v_e) while Equation 11 calculates the mass flow rate ($\dot{m}$).

$$v_e = I_{sp} g_0 \qquad \text{Eq. 10}$$

$$\dot{m} = \frac{F_{thrust}}{v_e} \qquad \text{Eq. 11}$$

To check for accuracy, the time the third stage burns can be calculated and compared with the reported burn time. The gross mass and the empty mass of the third stage are needed for this calculation. The fuel burned (m) is found with Equation 12, and the burn time (t) is calculated using Equation 13.

$$m = m_3 - m_{3,e} \quad \text{Eq. 12}$$

$$m = \dot{m}t \quad \text{Eq. 13}$$

This results in a burn time of 438 seconds which is consistent with the reported burn time of 500 seconds and has an error of 12.4%. Next, the initial and final mass of the Apollo 11 spacecraft before and after the trans-lunar injection burn will be calculated. This involves the total mass of the Saturn V, the gross mass of all the stages, and the mass of the fuel burned in the first third stage burn. To find the mass of fuel burned in the first third stage burn (m_{burnt}), the ratio of the first burn duration to the total burn duration will be multiplied by the mass of the total fuel. This is calculated in Equation 14.

$$m_{burnt} = \frac{t_1}{t_{total}} m \quad \text{Eq. 14}$$

Now the terms m_f and m_0 of Equation 1 can be calculated with m_{burnt} , the total mass of the Saturn V, the gross mass of the stages, the mass of the launch escape, and the mass of the usable fuel. The mass of the usable fuel ($m_{useable}$) is calculated by Equation 15.

$$m_{useable} = m - m_{burnt} \quad \text{Eq. 15}$$

The terms m_f and m_0 of Equation 1 are calculated as follows:

$$m_0 = m_{Total} - m_1 - m_2 - m_{burnt} - m_{se} \quad \text{Eq. 16}$$

$$m_f = m_0 - m_{useable} \quad \text{Eq. 17}$$

The Δv for the trans-lunar injection burn can be calculated with the terms m_0, m_f, and v_e with Equation 1. This results in a Δv of 2,937 meters per second, which is consistent with the prior Δv calculation of 3,129 meters per second. The difference is 6.5%.

IV. Results

The results of the previous calculations are shown in Table 4 alongside the figures reported by NASA and the relative errors.

Table 4. Velocity Calculations and Error

Calculated Quantity	Calculation	Reported Value	Error
Earth Orbital Speed	7791 m/s	7823 m/s	0.41%
Trans-lunar Injection Speed	10926 m/s	10952 m/s	0.24%
Delta-v (Orbital Mechanics)	3135 m/s	3129 m/s	0.19%
Delta-v (Rocket Equation)	2937 m/s	3129 m/s	6.14%

V. MATLAB Model

The previous model analytically approximates the actual trans-lunar trajectory by neglecting the gravitational influence of the Moon. Given the comparatively small gravitational influence of the Moon relative to the Earth, this is a reasonable approximation. However, a numerical solution can provide a more accurate approximation for the two-body system. With such a model, the MATLAB interface can be used to create a visual simulation of the trajectory. The numerical model in this report will use the Improved Euler and fourth order Runge Kutta methodologies. The MATLAB script is included in Appendix B.

If a two-dimensional Cartesian coordinate system is imposed on the two-body system where the center of the Earth lies at the origin, and the movement of the Earth is assumed to be negligible, the x and y coordinates of the spacecraft can be represented as x and y. Similarly, the x and y coordinates of the Moon can be represented x_m and y_m. Using Newton's Law of Universal Gravitation with two bodies and constants in Table 3, the acceleration of the spacecraft in the x and y directions, noted a_x and a_y, can be computed as follows:

$$a_x = \frac{\mu_m (x_m - x)}{((x_m - x)^2 + (y_m - y)^2)^{3/2}} - \frac{\mu_E x}{(x^2 + y^2)^{3/2}} \qquad \text{Eq. 18}$$

$$a_y = \frac{\mu_m (y_m - y)}{((x_m - x)^2 + (y_m - y)^2)^{3/2}} - \frac{\mu_E y}{(x^2 + y^2)^{3/2}} \qquad \text{Eq. 19}$$

While a_x and a_y represent the computed values of the spacecraft's acceleration, $f(x, y)$ and $g(y, x)$ will be used to represent the expressions for calculating the x and y accelerations respectively.

Given the initial conditions for the position and velocity of the spacecraft, the Runge Kutta method can be used to make iterative calculations of the velocity using iterative calculations of a_x and a_y. The counting index for the iterations will be represented as n and the step size as Δt. The x and y velocities of the spacecraft will be noted v_x and v_y, and are calculated as follows:

$$v_{x,n+1} = v_{x,n} + \frac{1}{6}(k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x}) \qquad \text{Eq. 20}$$

$$v_{y,n+1} = v_{y,n} + \frac{1}{6}(k_{1y} + 2k_{2y} + 2k_{3y} + k_{4y}) \qquad \text{Eq. 21}$$

Where,

$$k_{1x} = \Delta t(f(x_n, y_n)) \qquad k_{1y} = \Delta t(g(y_n, x_n))$$

$$k_{2x} = \Delta t\left(f\left(x_n + \frac{\Delta t}{2}, y_n + \frac{k_{1x}}{2}\right)\right) \qquad k_{2y} = \Delta t\left(g\left(y_n + \frac{\Delta t}{2}, x_n + \frac{k_{1y}}{2}\right)\right)$$

$$k_{3y} = \Delta t\left(g\left(y_n + \frac{\Delta t}{2}, x_n + \frac{k_{2y}}{2}\right)\right) \qquad k_{3y} = \Delta t\left(g\left(y_{t-1} + \frac{\Delta t}{2}, x_{t-1} + \frac{k_{2y}}{2}\right)\right)$$

$$k_{4x} = \Delta t(f(x_n + \Delta t, y_n + k_{3x})) \qquad k_{4y} = \Delta t(g(y_n + \Delta t, x_n + k_{3y}))$$

Finally, x and y can be calculated with v_x and v_y using Improved Euler's Method.

$$x_{n+1} = x_n + \frac{\Delta t}{2}(v_{x,n} + v_{x,n+1}) \quad \text{Eq. 22}$$

$$y_{n+1} = y_n + \frac{\Delta t}{2}(v_{y,n} + v_{y,n+1}) \quad \text{Eq. 23}$$

Animating the plot of the position of the spacecraft with MATLAB yields Figures 2 and 3. The trajectory of the spacecraft is marked with green, and the Earth and the Moon are drawn in blue and black respectively. The spacecraft orbits the Earth once before initiating the burn for trans-lunar injection and completing a free-return trajectory. The Moon is not drawn to scale.

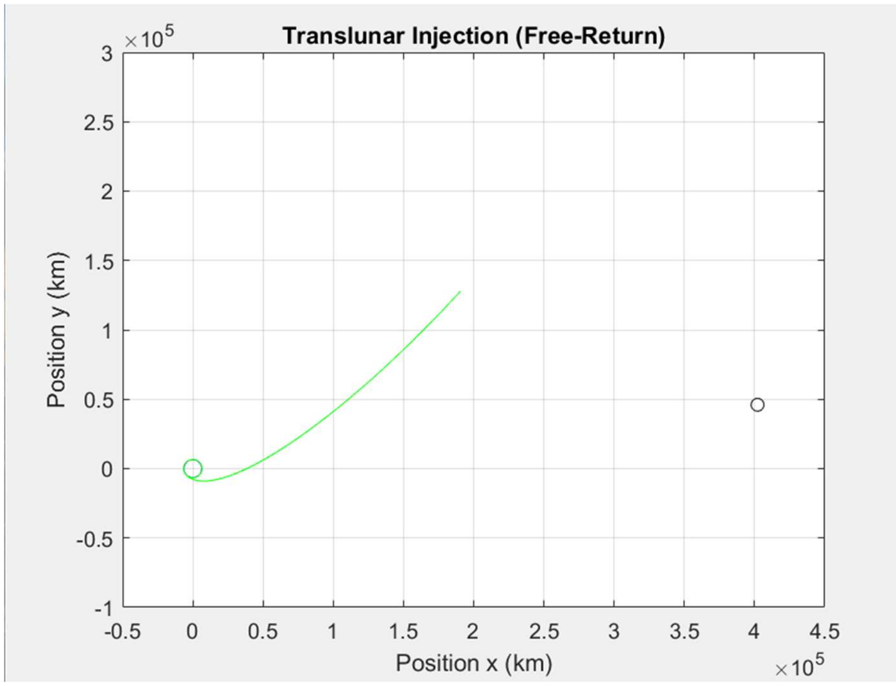

Figure 2. Trans-lunar Insertion in Progress

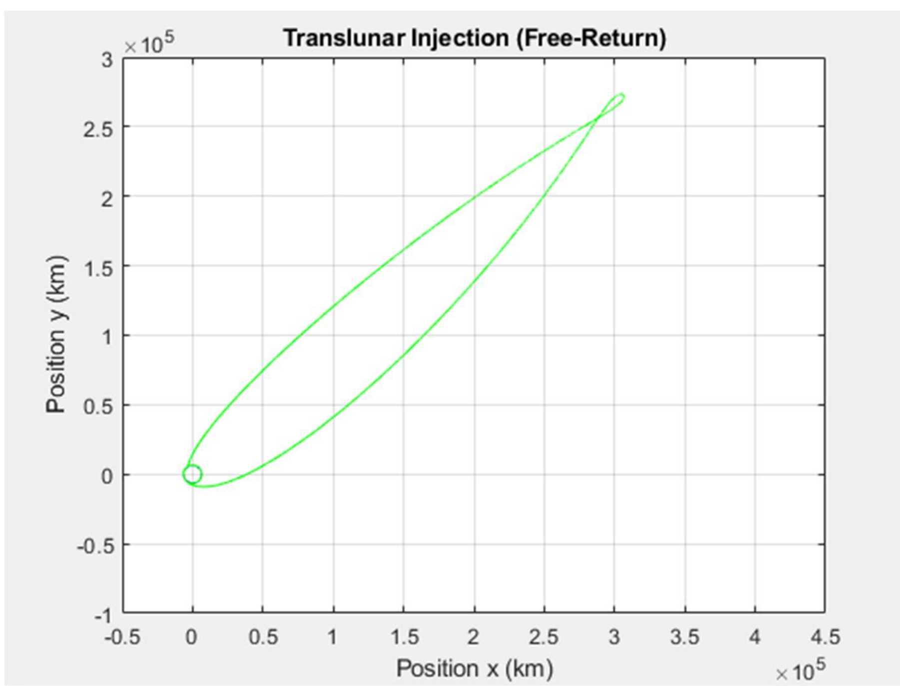

Figure 3. Complete Free-Return Path

VI. Conclusion

As demonstrated, a basic understanding of orbital mechanics and the equations governing rocket propulsion yields a reasonably accurate model of a complex maneuver such as the trans-lunar injection of Apollo 11. Although the analytical model is only a rough approximation, the ease of implementing numerical methods with modern computing allows for precise solutions to complicated non-linear differential equations. The MATLAB software and interface allows for these solutions to be computed with relatively simple code and simulated in a visually comprehensive way. In this modeling, the math involved in rocket propulsion and orbital mechanics illustrates the importance of differential equations and their applications in physical systems. Today, advancements in aerospace and astronautics are being made at a surprising rate and have inspired a new interest in space exploration that has not been felt since the Apollo program. These advancements all are dependent on mathematical modeling using differential equations and numerical solutions.

Appendix A

Calculations for r_p and r_a values (Eq.2, Eq.3):

$$r_p = 6.371\times10^6 m + 185075m = 6556075m$$

$$r_a = 1.737\times10^6 m + 106619m = 1823619m$$

Calculation for Apollo 11 orbital velocity (v_0 , Eq. 5):

$$v_0 = \sqrt{\frac{3.98\times10^{14}\frac{m^3}{s^2}}{6556075m}} = 7791\frac{m}{s}$$

Calculation for Equation 6:

$$2a = 6556075m + 1823619m + 3.844\times10^8 m$$

$$2a = 392779694m$$

Calculation for trans-lunar injection velocity (v_f , Eq. 8):

$$v_f = \sqrt{2\left(\frac{3.98\times10^{14}\frac{m^3}{s^2}}{6556075m} - \frac{3.98\times10^{14}\frac{m^3}{s^2}}{392779694m}\right)}$$

$$v_f = 10926\frac{m}{s}$$

Calculation for orbital mechanics Δv (Eq.12):

$$\Delta v = 10926\frac{m}{s} - 7791\frac{m}{s} = 3135\frac{m}{s}$$

Calculation for effective exhaust velocity (v_e , Eq. 10):

$$v_e = 421s\left(9.8\frac{m}{s^2}\right) = 4125\frac{m}{s}$$

Calculation for mass flow rate ($\dot{m}$, Eq. 11):

$$\dot{m} = \frac{1.0331\times10^6 N}{4125\frac{m}{s}} = 250\frac{kg}{s}$$

Calculations for Equations 12 and 13:

$$m = 123000kg - 13500kg = 109500kg$$

$$t = \frac{m}{\dot{m}} = \frac{109500kg}{250\frac{kg}{s}} = 438s$$

Calculation for mass burnt in first third stage burn (m_{burnt} , Eq. 14):

$$m_{burnt} = \frac{165s}{500s}(109500kg) = 36135kg$$

Calculation for useable third stage fuel mass ($m_{useable}$, Eq. 15):

$$m_{useable} = 109500kg - 36135kg = 73365kg$$

Calculation for Apollo 11 mass pre-burn (m_0 , Eq. 16):

$$m_0 = 2.97 \times 10^6 kg - 2.29 \times 10^6 kg - 496200kg - 36135kg - 3631kg$$

$$m_0 = 144034kg$$

Calculation for Apollo 11 mass post-burn (m_f , Eq. 17):

$$m_f = 144034kg - 73365kg = 70669kg$$

Calculation for rocket equation Δv (Eq. 1):

$$\Delta v = 4125s\left(\ln\left(\frac{144034kg}{70669kg}\right)\right) = 2937\frac{m}{s}$$

Appendix B

MATLAB function files:

```
function apollo_accelx = Apollo_accelf(x,y,mx,my)
  G_Param = 5.162387e12; % km^3/hr^2
  G_Param_Moon = 6.351564e10; % km^3/hr^2
  dt = 0.01;
  dt1 = 0.005;

  k1 = dt*((G_Param_Moon*(mx-x))/(((mx-x)^2 + (my-y)^2)^(1.5)) -
(G_Param*x)/((x^2 + y^2)^(1.5)));
  k2 = dt*((G_Param_Moon*(mx-(x+dt1)))/(((mx-(x+dt1))^2 + (my-
(y+0.5*k1))^2)^(1.5)) - (G_Param*(x+dt1))/(((x+dt1)^2 +
(y+0.5*k1)^2)^(1.5)));
  k3 = dt*((G_Param_Moon*(mx-(x+dt1)))/(((mx-(x+dt1))^2 + (my-
(y+0.5*k2))^2)^(1.5)) - (G_Param*(x+dt1))/(((x+dt1)^2 +
(y+0.5*k2)^2)^(1.5)));
  k4 = dt*((G_Param_Moon*(mx-(x+dt)))/(((mx-(x+dt))^2 + (my-(y+k3))^2)^1.5) -
(G_Param*(x+dt))/(((x+dt)^2 + (y+k3)^2)^(1.5)));

  apollo_accelx = (1/6)*(k1 + 2*k2 + 2*k3 + k4);
end

function moon_accel = moon_accelf(mx,my,d)
  G_Param = 5.162387e12; % km^3/hr^2
  moon_accel = -(G_Param*d)/(((mx^2 + my^2))^(1.5));
end
```

Simulation Script:

```
% Translunar Insertion Simulation (Newtonian)
% Runge Kutta and Improved Euler's

clear
clc

%% Constants

Earth_Radius = 6371; % km
Moon_Radius = 1737; % km
Moon_Apogee = 405696; % km
Moon_Perigee = 363104; % km
G_Param = 5.162387e12; % km^3/hr^2
G_Param_Moon = 6.351564e10; % km^3/hr^2
Moon_Period = 696; % hrs
Moon_apogee_v = 3450; % km/hr
Apollo_Earth_alt = 185.075; % km
Apollo_Earth_orb_v = 28163; % km/hr
Apollo_Delta_v = 10750; % km/hr
Apollo_Delta_v2 = -4250; % km/hr
Burn_Time1 = 166; % iteration count of burn
Omega = Apollo_Earth_orb_v/(Apollo_Earth_alt+Earth_Radius); % rad/hr
```

```
Moon_Initial_Angle = -0.15; % rad

%% Earth Drawing

Earth_x = linspace(-Earth_Radius, Earth_Radius, 1000);
Earth_y = [sqrt(Earth_Radius^2 - (Earth_x).^2);-sqrt(Earth_Radius^2 -
(Earth_x).^2)];

%% Lunar Orbit Initial Conditions

%Iteration Specifics
dt=0.01; % 36 seconds
index = 0:dt:0.33*Moon_Period; % iteration counter

% Position
Moon_pos = zeros(2,length(index)); % row 1 = x / row 2 = y
Moon_pos(1:2,1) =
[Moon_Apogee*cos(Moon_Initial_Angle);Moon_Apogee*sin(Moon_Initial_Angle)];

% Velocity
Moon_v = zeros(2,length(index)); % row 1 = velocity in x / row 2 = velocity
in y
Moon_v(1:2,1) = [Moon_apogee_v*sin(-Moon_Initial_Angle);Moon_apogee_v*cos(-
Moon_Initial_Angle)];

% Acceleration
Moon_a = zeros(2,length(index)); % row 1 = accel in x / row 2 = accel in y
Moon_a(1:2,1) =
[moon_accelf(Moon_pos(1,1),Moon_pos(2,1),Moon_pos(1,1));moon_accelf(Moon_pos(
1,1),Moon_pos(2,1),Moon_pos(2,1))];

%% Apollo Spacecraft Initial Conditions

% Position
Apollo_pos = zeros(2,length(index)); % row 1 = x / row 2 = y
Apollo_pos(1:2,1) = [-(Apollo_Earth_alt + Earth_Radius);0];

% Velocity
Apollo_v = zeros(3,length(index)); % row 1 = velocity in x / row 2 = velocity
in y / row 3 = velocity in y
Apollo_v(1:3,1) = [0;-Apollo_Earth_orb_v;Apollo_Earth_orb_v];

% Acceleration
Apollo_a=zeros(3,length(index)); % row 1 = accel in x / row 2 = accel in y /
row 3 = total accel
Apollo_a(1:2,1) =
[Apollo_accelf(Apollo_pos(1,1),Apollo_pos(2,1),Moon_pos(1,1),Moon_pos(2,1));A
pollo_accelf(Apollo_pos(2,1),Apollo_pos(1,1),Moon_pos(2,1),Moon_pos(1,1))];
Apollo_a(3,1) = sqrt(Apollo_a(1,1)^2 + Apollo_a(2,1)^2);

%% Simulation

for t=2:length(index)

    % Moon Acceleration
```

```
    Moon_a(1:2,t) = [moon_accelf(Moon_pos(1,t-1),Moon_pos(2,t-
1),Moon_pos(1,t-1));moon_accelf(Moon_pos(1,t-1),Moon_pos(2,t-1),Moon_pos(2,t-
1))];

    % Moon Velocity
    Moon_v(1:2,t) = [Moon_v(1,t-1) + 0.5*dt*(Moon_a(1,t-
1)+Moon_a(1,t));Moon_v(2,t-1) + 0.5*dt*(Moon_a(2,t-1)+Moon_a(2,t))];

    % Moon Position
    Moon_pos(1:2,t) = [Moon_pos(1,t-1) + 0.5*dt*(Moon_v(1,t-
1)+Moon_v(1,t));Moon_pos(2,t-1) + 0.5*dt*(Moon_v(2,t-1)+Moon_v(2,t))];

    % Apollo Acceleration
    Apollo_a(1:2,t) = [Apollo_accelf(Apollo_pos(1,t-1),Apollo_pos(2,t-
1),Moon_pos(1,t-1),Moon_pos(2,t-1));Apollo_accelf(Apollo_pos(2,t-
1),Apollo_pos(1,t-1),Moon_pos(2,t-1),Moon_pos(1,t-1))];
    Apollo_a(3,t) = sqrt(Apollo_a(1,t)^2 + Apollo_a(2,t)^2);

    % Apollo Velocity
    if t==Burn_Time1 % delta-v

        angle = atan(Apollo_v(2,t-1)/Apollo_v(1,t-1));
            if Apollo_v(2,t-1)<0 && Apollo_v(1,t-1)<0 || Apollo_v(2,t-1)>0 &&
Apollo_v(1,t-1)<0
                angle = angle + pi;
            else
            end
        Apollo_v(1:2,t) = [(Apollo_Earth_orb_v +
Apollo_Delta_v)*cos(angle);(Apollo_Earth_orb_v + Apollo_Delta_v)*sin(angle)];
        Apollo_v(3,t) = sqrt(Apollo_v(1,t)^2 + Apollo_v(2,t)^2);


    elseif t>Burn_Time1 % Normal rk

        Apollo_v(1:2,t) = [Apollo_v(1,t-1) + Apollo_a(1,t);Apollo_v(2,t-1) +
Apollo_a(2,t)];
        Apollo_v(3,t) =  sqrt(Apollo_v(1,t)^2 + Apollo_v(2,t)^2);

    else % Earth Orbit

        Apollo_v(1:2,t) = [-
Apollo_Earth_orb_v*sin(Omega*index(t)+pi);Apollo_Earth_orb_v*cos(Omega*index(
t)+pi)];
        Apollo_v(3,t) = sqrt(Apollo_v(1,t)^2 + Apollo_v(2,t)^2);

    end

    % Apollo Position
    if t<=Burn_Time1 % Earth Orbit
        Apollo_pos(1:2,t) =
[(Apollo_Earth_alt+Earth_Radius)*cos(Omega*index(t)+pi);(Apollo_Earth_alt+Ear
th_Radius)*sin(Omega*index(t)+pi)];
    else
        if ((Moon_pos(1,t-1)-Apollo_pos(1,t-1))^2  + (Moon_pos(2,t-1)-
Apollo_pos(2,t-1))^2) < (Moon_Radius)^2
```

```
            Apollo_pos(1:2,t) = [Moon_pos(1,t-1); Moon_pos(2,t-1)]; % Apollo
hits moon

        elseif Apollo_pos(1,t-1)^2 + Apollo_pos(2,t-1)^2 < Earth_Radius^2

            Apollo_pos(1:2,t) = [Apollo_pos(1,t-1);Apollo_pos(2,t-1)]; %
Apollo hits Earth

        else

            Apollo_pos(1:2,t) = [Apollo_pos(1,t-1) + 0.5*dt*(Apollo_v(1,t-
1)+Apollo_v(1,t));Apollo_pos(2,t-1) + 0.5*dt*(Apollo_v(2,t-
1)+Apollo_v(2,t))];

        end
    end

    % Earth
    plot(Earth_x, Earth_y(1,:),'b', Earth_x, Earth_y(2,:),'b')
    grid on
    axis([-50000 450000 -100000 300000])
    hold on

    % Simulation Plot

plot(Moon_pos(1,t),Moon_pos(2,t),'ko',Apollo_pos(1,1:t),Apollo_pos(2,1:t),'g'
)
    title('Translunar Injection (Free-Return)')
    xlabel('Position x (km)')
    ylabel('Position y (km)')

    pause(0.0001);
    hold off

end
```

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